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ON MULTIPLE CIRCULAR INCLUSIONS IN PLANE THERMOELASTICITY

C. K. CHAO, M. H. SHEN and C. K. FUNG
Department of Mechanical Engineering, National Taiwan Institute of Technology, Taipei, Taiwan, 106, Republic of China

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Abstract—A general series solution to the problem of interacting circular inclusions in plane thermoelasticity is provided in this paper. Based upon the complex variable theory and the use of Laurent series expansion, the general expression of the stress functions is derived explicitly for the circular inclusion problem under remote uniform heat flow. By applying the use of the superposition, the problem dealing with any number of arbitrarily located inclusions can be then reduced to a set of linear algebraic equations which are solved with the aid of a perturbation technique. For illustrating the use of the present approach, an approximate closed form solution of the stress functions is derived explicitly for the problem containing two arbitrarily located inclusions. Numerical results of the interfacial stresses around a rigid circular inclusion or hoop stress along a circular hole due to the presence of an elastic inclusion are provided to demonstrate the dependence of the solution upon the pertinent parameters. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

Studies on interacting inclusions or inhomogeneities in composite materials have been a topic of considerable research. For example, determination of the stress fields induced by an infinite number of periodically or randomly distributed inclusions has been obtained by Hashin (1983), Willis (1983) and Isida and Igawa (1991). The elasticity problems associated with a finite number of arbitrarily located inclusions were considered by Moschovidis and Mura (1975), Tandon and Weng (1986), Rodin and Hwang (1991) and Gong and Meguid (1993). The elastic field for two circular inclusions in antiplane elastostatics has been solved by Goree and Wilson (1967), Budiansky and Carrier (1984), Steif (1989) and Honein *et al.* (1992). All the afore-mentioned studies have concentrated on multiple inclusion problems under isothermal loading conditions. Now a challenging problem is to find the stress field of the current problem of arbitrarily located inclusions subjected to thermal loadings. This problem, to the authors' knowledge, has not been considered in the literature.

The problem associated with one single inclusion perfectly bonded to an infinite matrix subjected to arbitrary thermal loadings was recently solved by Chao and Shen (1995). The general exact solutions of the thermal stress field in both the inclusion and the surrounding matrix are obtained by using the method of analytical continuation. However, the above-mentioned methodology cannot be directly used to solve the present problem of multiple inclusions with two or more separate interfaces that a closed-form solution is impossible to achieve. In this paper, both the complex variable theory and the Laurent series expansion [Isida (1973); Gong and Meguid (1993)] will be used to derive the general expressions of the complex potentials which satisfy the prescribed continuity conditions for each circular inclusion. For studying the interaction effects among the inclusions, the superposition principle is applied to reduce the problem to a set of linear algebraic equations. Special examples of two circular inclusions embedded in an infinite matrix under remote uniform heat flow are given to illustrate the use of the present approach. By using a perturbation technique, an approximate closed-form solution up to the fourth order of the stress functions in the matrix is presented explicitly. For a limiting case when two circular inclusions are sufficiently apart, the zero-order solution of the stress functions derived in this work is shown to coincide exactly with the result of the corresponding single inclusion problem. Numerical examples concerned with either a circular hole or a rigid circular inclusion interacted with an elastic inclusion are provided to demonstrate the dependence of the normalized hoop stress or interfacial stresses upon the material parameters and the relative position of the inclusion. The results presented here will be helpful in understanding the thermoelastic interaction behavior when two circular inclusions become close to each other.

2. PROBLEM FORMULATION

For two-dimensional steady-state heat conduction problems, the temperature T and the total heat flow Q can be given in terms of an analytic function $g'(z)$ of a complex variable $z = x+iy$, namely

$$T = \mathrm{Re}\,[g'(z)] \tag{1}$$

$$Q = -k\,\mathrm{Im}\,[g'(z)] \tag{2}$$

where k is the heat conductivity and Re and Im stand for the real part and imaginary part of the argument, respectively. The components of the displacement and traction force, in two-dimensional theory of thermoelasticity, can be expressed in terms of two stress functions $\phi(z)$, $\psi(z)$ and a temperature function $g'(z)$ as

$$2\mu(u+iv) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} + 2\mu\beta\int g'(z)\,\mathrm{d}z \tag{3}$$

$$-Y+iX = \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)} \tag{4}$$

where μ is the shear modulus and

$$\kappa = \frac{3-\nu}{1+\nu}, \quad \beta = \alpha$$

for plane stress and $\kappa = 3-4\nu$, $\beta = (1+\nu)\alpha$ for plane strain with ν being the Poisson's ratio and α the thermal expansion coefficient. Primes denote differentiation with respect to z and a superimposed bar denotes the complex conjugate. Consider an array of circular inclusions, of arbitrary radii a_j and of different shear moduli μ_j and heat conductivities k_j, perfectly bonded to a matrix, of infinite extent and of shear modulus μ and heat conductivity k, subjected to remote uniform heat flux q_0 (see Fig. 1). We now seek the solution of the circular inclusion problem as the form

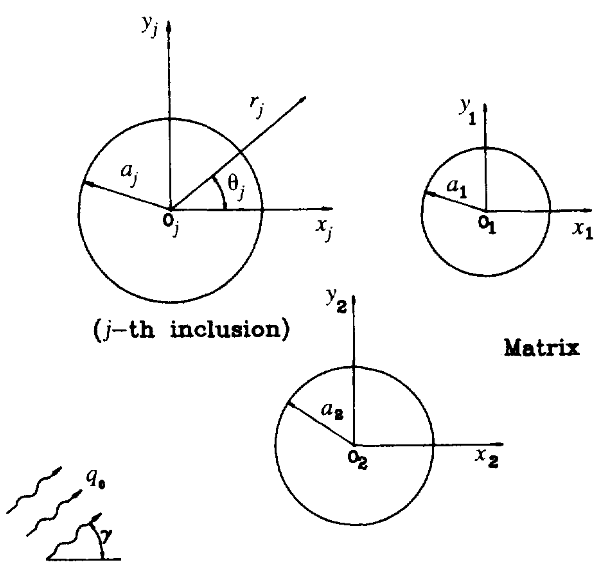


Fig. 1. Circular inclusions in an isotropic thermoelastic medium.

If the matrix and the circular inclusions are assumed to be perfectly bonded along the interface $z_j = a_j e^{i\theta_j}$, both the continuity conditions $T = T_j$ and $Q = Q_j$ across the interface lead to

$$\sum_{n=0}^{\infty} \{C_{n,j}a_j^{n+1}z_0^{n+1} + \bar{C}_{n,j}a_j^{n+1}z_0^{-(n+1)} + D_{n,j}a_j^{-(n+1)}z_0^{-(n+1)} + \bar{D}_{n,j}a_j^{-(n+1)}z_0^{n+1}\}$$

$$= \sum_{n=0}^{\infty} \{E_{n,j}a_j^{n+1}z_0^{n+1} + \bar{E}_{n,j}a_j^{n+1}z_0^{-(n+1)}\} \quad (16)$$

and

$$\sum_{n=0}^{\infty} k\{C_{n,j}a_j^{n+1}z_0^{n+1} - \bar{C}_{n,j}a_j^{n+1}z_0^{-(n+1)} + D_{n,j}a_j^{-(n+1)}z_0^{-(n+1)} - \bar{D}_{n,j}a_j^{-(n+1)}z_0^{n+1}\}$$

$$= \sum_{n=0}^{\infty} k_j\{E_{n,j}a_j^{n+1}z_0^{n+1} - \bar{E}_{n,j}a_j^{n+1}z_0^{-(n+1)}\} \quad (17)$$

where z_0 is represented as $e^{i\theta_j}$.

By comparing the coefficients of each power of z_0, the explicit relations among the coefficients are obtained as

$$D_{n,j} = \frac{k-k_j}{k+k_j} a_j^{2n+2} \bar{C}_{n,j} \quad (18)$$

$$E_{n,j} = \frac{2k}{k+k_j} C_{n,j}. \quad (19)$$

Equations (14)–(15) and eqns (18)–(19) constitute the general expressions of the complex potentials in the thermal field which automatically satisfy the continuity conditions for each circular inclusion. The only remaining unknown constant $C_{n,j}$ or $D_{n,j}$ will be determined once the specific geometry and loading conditions of the problem are given. Consider two arbitrarily located inclusions in an infinite matrix under remoted uniform heat flux (see Fig. 2). Let (x_j, y_j) and (r_j, θ_j) denote the rectangular Cartesian and polar coordinate systems with their origin O_j at the center of the jth inclusion. The quantity d_{jk} denotes the distance between the jth inclusion and the kth inclusion while φ_{jk} stands for the inclination angle measured from the x_j-axis to the O_jO_k. For an arbitrary array of N circular inclusions

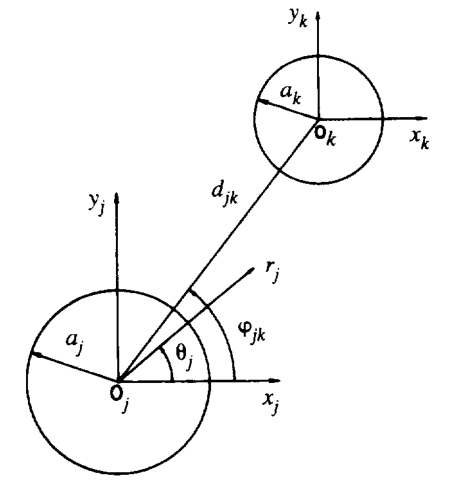


Fig. 2. Two arbitrarily located inclusions in an infinite matrix.

$$\phi(z_j) = A\log z_j + \phi^*(z_j) \tag{5}$$

$$\psi(z_j) = B\log z_j + \psi^*(z_j) \tag{6}$$

in the matrix and

$$\phi_j(z_j) = \phi_j^*(z_j) \tag{7}$$

$$\psi_j(z_j) = \psi_j^*(z_j) \tag{8}$$

in circular inclusions.

Since no singularities are assumed to reside inside or on the boundary of the inclusion, $\phi^*(z_j)$, $\psi^*(z_j)$ and $\phi_j^*(z_j)$, $\psi_j^*(z_j)$ can be, respectively, expanded into Laurent series and Taylor series as follows

$$\phi^*(z_j) = \sum_{n=0}^{\infty} [M_{n,j} z_j^{n+1} + F_{n,j} z_j^{-(n+1)}] \tag{9}$$

$$\psi^*(z_j) = \sum_{n=0}^{\infty} [K_{n,j} z_j^{n+1} + S_{n,j} z_j^{-(n+1)}] \tag{10}$$

and

$$\phi_j^*(z_j) = \sum_{n=0}^{\infty} H_{n,j} z_j^{n+1}, \quad \psi_j^*(z_j) = \sum_{n=0}^{\infty} L_{n,j} z_j^{n+1} \tag{11}$$

It should be noted that the singular term $\log z_j$ appearing in eqns (5) and (6) results from the requirement of single-valued conditions applying each circular inclusion. The constants A, B must satisfy the following equations [Chao and Shen (1995)]

$$\kappa A + \bar{B} = \frac{-2\mu\beta}{2\pi i}[g(z_j)]_{c_j} \tag{12}$$

$$A - \bar{B} = \frac{1}{2\pi i}[(-Y + iX)]_{c_j} \tag{13}$$

where $[f(z)]_{c_j}$ denotes the jump in $f(z)$ for a contour c_j. The remaining unknown constants appeared in eqns (9)–(11) may be determined from the interface continuity conditions.

3. COMPLEX POTENTIALS IN THERMAL FIELD

Since no singularities are assumed to reside inside or on the boundary of the inclusion, the temperature functions can be, respectively, represented as

$$g'(z_j) = \sum_{n=0}^{\infty} [C_{n,j} z_j^{n+1} + D_{n,j} z_j^{-(n+1)}] \tag{14}$$

in the matrix and

$$g_j'(z_j) = \sum_{n=0}^{\infty} E_{n,j} z_j^{n+1} \tag{15}$$

in the circular inclusion.

located in an infinite matrix under remote uniform heat flux, the temperature function in the matrix can be put in the form

$$g'(z_j) = \tau e^{-i\gamma} z_j + \sum_{k=1}^{N} \sum_{n=0}^{\infty} D_{n,k} z_k^{-(n+1)} \tag{20}$$

where $\tau = -q_0/k$. It should be noted that eqn (20) already satisfies the condition applied at infinity and the continuity conditions along the inclusion boundaries. After having the relation between the jth and the kth inclusion coordinates, i.e.

$$z_k = z_j - d_{jk} e^{i\varphi_{jk}} \tag{21}$$

Equation (20) can be rearranged as the form

$$g'(z_j) = \sum_{n=0}^{\infty} [C_{n,j} z_j^{n+1} + D_{n,j} z_j^{-(n+1)}] \tag{22}$$

where

$$C_{n,j} = \tau e^{-i\gamma} \delta_{0,n} + \sum_{p=0}^{\infty} \sum_{k \neq j}^{N} a_{n,j}^{p,k} D_{p,k} \tag{23}$$

with $\delta_{0,n}$ being the Kronerker delta and

$$a_{n,j}^{p,k} = \frac{(-1)^{p+1}}{(d_{jk})^{n+p+2}} \binom{n+p+1}{p} e^{-(n+p+2)\varphi_{jk} i} \tag{24}$$

By substituting eqn (18) into eqn (23), the unknown constant $D_{n,j}$ could be obtained and the thermal problem associated with N circular inclusions is thus solved.

4. COMPLEX POTENTIALS IN PLANE THERMOELASTICITY

By integrating the temperature function from eqn (22) and knowing that no resultant force is applied on any contour c_j, the constants A and B in eqns (12) and (13) yield the following results [Chao and Shen (1995)]

$$A = \frac{-2\mu\beta D_{0,j}}{1+\kappa}, \quad B = \frac{-2\mu\beta \bar{D}_{0,j}}{1+\kappa}. \tag{25}$$

The stress functions in the matrix and in the inclusion, respectively, now become

$$\phi(z_j) = \frac{-2\mu\beta D_{0,j}}{1+\kappa} \log z_j + \sum_{n=0}^{\infty} [M_{n,j} z_j^{n+1} + F_{n,j} z_j^{-(n+1)}] \tag{26}$$

$$\psi(z_j) = \frac{-2\mu\beta \bar{D}_{0,j}}{1+\kappa} \log z_j + \sum_{n=0}^{\infty} [K_{n,j} z_j^{n+1} + S_{n,j} z_j^{-(n+1)}] \tag{27}$$

and

$$\phi_j(z_j) = \sum_{n=0}^{\infty} H_{n,j} z_j^{n+1}, \quad \psi_j(z_j) = \sum_{n=0}^{\infty} L_{n,j} z_j^{n+1}. \tag{28}$$

If the circular inclusions and the matrix are assumed to be perfectly bonded along the interface, the displacements and surface tractions at the interface must be continuous. By using the general solutions (26)–(28), both the traction and displacement continuity conditions lead to

$$\begin{aligned}\sum_{n=0}^{\infty} \{ & M_{n,j} a_j^{n+1} z_0^{n+1} + F_{n,j} a_j^{-(n+1)} z_0^{-(n+1)} + \bar{A} z_0^2 + (n+1)\bar{M}_{n,j} a_j^{n+1} z_0^{-n+1} \\ & - (n+1)\bar{F}_{n,j} a_j^{-(n+1)} z_0^{n+3} + \bar{K}_{n,j} a_j^{n+1} z_0^{-(n+1)} + \bar{S}_{n,j} a_j^{-(n+1)} z_0^{n+1} \} \\ & = \sum_{n=0}^{\infty} \{ H_{n,j} a_j^{n+1} z_0^{n+1} + (n+1)\bar{H}_{n,j} a_j^{n+1} z_0^{-n+1} + \bar{L}_{n,j} a_j^{n+1} z_0^{-(n+1)} \}\end{aligned} \tag{29}$$

$$\begin{aligned}\sum_{n=0}^{\infty} \Big\{ & \Gamma_j [\kappa M_{n,j} a_j^{n+1} z_0^{n+1} + \kappa F_{n,j} a_j^{-(n+1)} z_0^{-(n+1)} - \bar{A} z_0^2 - (n+1)\bar{M}_{n,j} a_j^{n+1} z_0^{-n+1} \\ & + (n+1)\bar{F}_{n,j} a_j^{-(n+1)} z_0^{n+3} - \bar{K}_{n,j} a_j^{n+1} z_0^{-(n+1)} - \bar{S}_{n,j} a_j^{-(n+1)} z_0^{n+1}] \\ & + 2\beta\mu_j \left[\frac{1}{n+2} C_{n,j} a_j^{n+2} z_0^{n+2} - \frac{1}{n+1} D_{n,j} a_j^{-(n+1)} z_0^{-(n+1)} \right] \Big\} \\ = \sum_{n=0}^{\infty} \{ & H_{n,j} a_j^{n+1} z_0^{n+1} - (n+1)\bar{H}_{n,j} a_j^{n+1} z_0^{-n+1} - \bar{L}_{n,j} a_j^{n+1} z_0^{-(n+1)} \\ & + 2\beta_j \mu_j \frac{1}{n+2} E_{n,j} a_j^{n+2} z_0^{n+2} \}\end{aligned} \tag{30}$$

where $\Gamma_j = \mu_j/\mu$. By comparing the coefficients of each power of z_0, the explicit results among the coefficients of the complex potentials are obtained as

$$S_{0,j} = -2\gamma_j \operatorname{Re}[M_{0,j}] a_j^2 \tag{31}$$

$$S_{1,j} = -\alpha_j a_j^4 \bar{M}_{1,j} + \frac{\mu_j(\beta\bar{C}_{0,j} - \beta_j \bar{E}_{0,j})}{\Gamma_j + \kappa_j} a_j^4 - A a_j^2 \tag{32}$$

$$\begin{aligned} S_{n,j} = & -[(n^2-1)\delta_j + \alpha_j] a_j^{2n+2} \bar{M}_{n,j} - (n-1)\delta_j \bar{K}_{n-2,j} a_j^{2n} \\ & + \frac{2\mu_j \beta \bar{D}_{n-1,j}}{1+\Gamma_j \kappa} a_j^2 + \frac{2\mu_j(\beta\bar{C}_{n-1,j} - \beta_j \bar{E}_{n-1,j})}{(n+1)(\Gamma_j + \kappa_j)} a_j^{2n+2}, \quad (n \geqslant 2) \end{aligned} \tag{33}$$

$$H_{0,j} = \frac{(1-\alpha_j)(1-\gamma_j\delta_j)}{1-\alpha_j\delta_j} M_{0,j} + \frac{(\alpha_j - \gamma_j)(1-\delta_j)}{1-\alpha_j\delta_j} \bar{M}_{0,j} \tag{34}$$

$$H_{n,j} = (1-\alpha_j) M_{n,j} + \frac{2\mu_j(\beta C_{n-1,j} - \beta_j E_{n-1,j})}{(n+1)(\Gamma_j + \kappa_j)}, \quad (n \geqslant 1) \tag{35}$$

$$\begin{aligned} L_{n,j} = & (1-\delta_j) K_{n,j} + (n+3)(\alpha_j - \delta_j) M_{n+2,j} a_j^2 + \frac{2\mu_j \beta \bar{D}_{n+1,j} a_j^{-(2n+2)}}{(n+1)(1+\Gamma_j\kappa)} \\ & - \frac{2\mu_j(\beta C_{n+1,j} - \beta_j E_{n+1,j}) a_j^2}{\Gamma_j + \kappa_j}, \quad (n \geqslant 0) \end{aligned} \tag{36}$$

$$F_{n,j} = -\delta_j \bar{K}_{n,j} a_j^{2n+2} - (n+3)\delta_j \bar{M}_{n+2,j} a_j^{2n+4} + \frac{2\mu_j \beta D_{n+1,j}}{(n+1)(1+\Gamma_j \kappa)}, \quad (n \geqslant 0) \tag{37}$$

where α_j, δ_j, γ_j are material constants for the jth inclusion defined as

$$\alpha_j = \frac{\kappa_j - \Gamma_j \kappa}{\Gamma_j + \kappa_j}, \quad \delta_j = \frac{1-\Gamma_j}{1+\Gamma_j \kappa}, \quad \gamma_j = \frac{\kappa_j - 1 - \Gamma_j(\kappa - 1)}{2\Gamma_j + (\kappa_j - 1)}. \tag{38}$$

Actually, there are only two independent combinations of the material moduli which are relevant, since we have

$$\gamma_j = \frac{\alpha_j - \delta_j}{(1-\delta_j) - \delta_j(1-\alpha_j)} \tag{39}$$

The remaining two unknown coefficients $M_{n,j}$, $K_{n,j}$ which appeared in eqns (31)–(37) could be obtained as the specific geometry and loading conditions of the problem are given. Equations (26)–(28) and eqns (31)–(37) now constitute the general expressions of the complex potentials in plane thermoelasticity which satisfy the continuity conditions for each circular inclusion. Note that the present expressions as described above can be directly reduced to the results of the thermoelastic problem associated with a single inclusion obtained by Chao and Shen (1995) and the isothermal elasticity problem associated with multiple inclusions obtained by Gong and Meguid (1993).

For the thermoelastic problem with an arbitrary array of N circular inclusions, the Airy' stress function in the matrix can be represented as

$$U = U_0 + \sum_{k=1}^{N} U_k \tag{40}$$

where U_0 represents the Airy' stress function corresponding to the homogeneous problem under the uniform stress state at infinity, i.e.

$$U_0 = \mathrm{Re}\left[\bar{z}_j \phi_0(z_j) + \int \psi_0(z_j)\, \mathrm{d}z_j\right] \tag{41}$$

with

$$\phi_0 = \tfrac{1}{4}(\sigma_x^\infty + \sigma_y^\infty) z_j \tag{42}$$

$$\psi_0 = \tfrac{1}{2}(\sigma_y^\infty - \sigma_x^\infty + 2i\tau_{xy}^\infty) z_j. \tag{43}$$

Since no mechanical loading is considered in the present study, i.e. $\sigma_x^\infty = \sigma_y^\infty = \tau_{xy}^\infty = 0$, it gives $\phi_0 = \psi_0 = U_0 = 0$. The Airy's stress functions U_k, which contains singularities inside the kth inclusion, are expressed as

$$U_k = \mathrm{Re}\,[\bar{z}_k \tilde{\phi}(z_k) + \int \tilde{\psi}(z_k)\, \mathrm{d}z_k] \tag{44}$$

where

$$\tilde{\phi}(z_k) = \sum_{n=0}^{\infty} F_{n,k} z_k^{-(n+1)} \tag{45}$$

$$\tilde{\psi}(z_k) = \sum_{n=0}^{\infty} S_{n,k} z_k^{-(n+1)}. \tag{46}$$

Substituting eqn (21) into eqn (44), the stress functions corresponding to eqn (40) can be rearranged as

$$\phi(z_j) = \sum_{n=0}^{\infty} [M_{n,j} z_j^{n+1} + F_{n,j} z_j^{-(n+1)}] \tag{47}$$

$$\psi(z_j) = \sum_{n=0}^{\infty} [K_{n,j} z_j^{n+1} + S_{n,j} z_j^{-(n+1)}] \tag{48}$$

where

$$M_{n,j} = \sum_{p=0}^{\infty} \sum_{k \neq j}^{N} a_{n,j}^{p,k} F_{p,k} \tag{49}$$

$$K_{n,j} = \sum_{p=0}^{\infty} \sum_{k \neq j}^{N} (a_{n,j}^{p,k} S_{p,k} + b_{n,j}^{p,k} F_{p,k}) \tag{50}$$

with

$$b_{n,j}^{p,k} = \frac{(-1)^{p+2}}{(d_{jk})^{n+p+2}} \binom{n+p+2}{p+1} e^{-(n+p+4)\varphi_{jk} i} \tag{51}$$

Equations (31)–(33), (37) and eqns (49)–(50) constitute the necessary conditions for determining the unknown coefficients $M_{n,j}$, $F_{n,j}$, $K_{n,j}$ and $S_{n,j}$ which will be solved successively by applying the perturbation technique. The coefficients $H_{n,j}$ and $L_{n,j}$ for the jth inclusion can then be determined through eqns (34)–(36). The thermoelastic problem of an infinitely extended matrix containing any number of arbitrarily located inclusions is thus solved.

5. RESULTS AND DISCUSSIONS

All the coefficients $C_{n,j}$, $D_{n,j}$ and $M_{n,j}$, $F_{n,j}$, $K_{n,j}$, $S_{n,j}$ ($n = 0, 1, 2, \ldots; j = 1, 2, \ldots$) appeared in eqn (22) and eqns (47)–(48), respectively, are solved successively in a Appendix by using a perturbation technique [Gong and Meguid (1993)]. In the present study, our attention will be focused on the change in the interfacial stresses at point A (see Fig. 3) around the first inclusion ($j = 1$) due to the presence of the second inclusion ($j = 2$). After having all

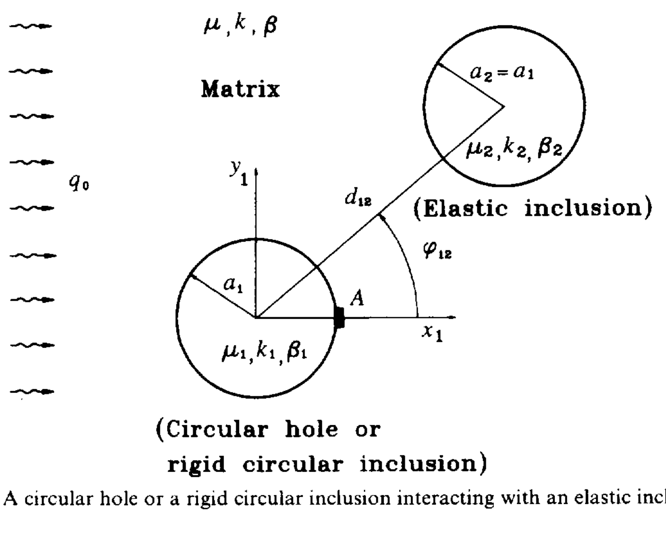


Fig. 3. A circular hole or a rigid circular inclusion interacting with an elastic inclusion.

the unknown coefficients, the stress complex potential in the matrix can be expressed as a series form

$$\Phi(z_j) = \phi'(z_j) = \sum_{n=0}^{\infty} \lambda^n \Phi^{(n)}(z_j) \tag{52}$$

$$\Psi(z_j) = \psi'(z_j) = \sum_{n=0}^{\infty} \lambda^n \Psi^{(n)}(z_j) \tag{53}$$

where λ is a perturbation parameter defined as a_1/d_{12}. Note that an approximate closed form solution in eqns (52) and (53) may be achieved since the stress functions $\Phi^{(n)}(z_j)$, $\Psi^{(n)}(z_j)$ can be determined successively to as many terms as required. When the two inclusions are sufficiently apart, the leading order terms of the series solution will provide an asymptotic solution to the given problem. In the following analysis, the solution up to the order of λ^4 is obtained explicitly as follows:

$$\Phi^{(0)}(z) = \frac{-2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)\tau a_1^2 e^{i\gamma} z^{-1} \tag{54}$$

$$\Psi^{(0)}(z) = \frac{-2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)\tau a_1^2 e^{-i\gamma} z^{-1} - 2\left[\frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right) + \frac{\mu_1}{\Gamma_1+\kappa_1}\left(\beta - \frac{2k}{k+k_1}\beta_1\right)\right] \tau e^{i\gamma} a_1^4 z^{-3} \tag{55}$$

$$\Phi^{(1)}(z) = \Psi^{(1)}(z) = 0 \tag{56}$$

$$\Phi^{(2)}(z) = \frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_1}\right)\left(\frac{a_2}{a_1}\right)^2 \tau a_1^2 e^{i(2\varphi_{12}-\gamma)} z^{-1} \tag{57}$$

$$\Psi^{(2)}(z) = \frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2 \tau a_1^2 e^{i(\gamma-2\varphi_{12})} z^{-1}$$
$$+2\left[\frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right) + \frac{\mu_1}{\Gamma_1+\kappa_1}\left(\beta - \frac{2k}{k+k_1}\beta_1\right)\left(\frac{k-k_2}{k+k_2}\right)\right]\left(\frac{a_2}{a_1}\right)^2 \tau a_1^4 e^{i(2\varphi_{12}-\gamma)} z^{-3} \tag{58}$$

$$\Phi^{(3)}(z) = 2\left[\frac{2\mu\beta\delta_1}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^4 + \frac{\mu_2\delta_1}{\Gamma_2+\kappa_2}\left(\beta - \frac{2k}{k+k_1}\beta_2\right)\left(\frac{a_2}{a_1}\right)^4\right.$$
$$\left. + \frac{\mu_1\beta}{1+\Gamma_1\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2\right] \tau e^{i(3\varphi_{12}-\gamma)} a_1^3 z^{-2} \tag{59}$$

$$\Psi^{(3)}(z) = 2\left[\frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right) + \frac{\mu_2}{\Gamma_2+\kappa_2}\left(\beta - \frac{2k}{k+k_2}\beta_2\right)\right]\left(\frac{a_2}{a_1}\right)^4 \tau a_1 e^{i(\gamma-3\varphi_{12})}$$
$$+2\left[\frac{6\mu\beta\delta_1}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^4 + \frac{3\mu_2\delta_1}{\Gamma_2+\kappa_2}\left(\beta - \frac{2k}{k+k_2}\beta_2\right)\left(\frac{a_2}{a_1}\right)^4\right.$$
$$+ \frac{3\mu_1\beta}{1+\Gamma_1\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2 + \frac{\mu_1}{\Gamma_1+\kappa_1}\left(\frac{k-k_2}{k+k_2}\right)\left(\beta - \frac{2k}{k+k_1}\beta_1\right)$$
$$\left. \times \left(\frac{a_2}{a_1}\right)^2\right] \tau a_1^5 e^{i(3\varphi_{12}-\gamma)} z^{-4} \tag{60}$$

$$\Phi^{(4)}(z) = \frac{-2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)^2\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2 \tau a_1^2\, e^{i\gamma} z^{-1}$$

$$+2\left[\frac{6\mu\beta\delta_1}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^4+\frac{3\mu_2\delta_1}{\Gamma_2+\kappa_2}\left(\beta-\frac{2k}{k+k_2}\beta_2\right)\left(\frac{a_2}{a_1}\right)^4\right.$$

$$\left.+\frac{\mu_1\beta}{1+\Gamma_1\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2\right]\tau a_1^4\, e^{i(4\varphi_{12}-\gamma)} z^{-3} \tag{61}$$

$$\Psi^{(4)}(z) = \frac{-2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)^2\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2 \tau a_1^2\, e^{-i\gamma} z^{-1}$$

$$+6\left[\frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right)+\frac{\mu_2}{\Gamma_2+\kappa_2}\left(\beta-\frac{2k}{k+k_2}\beta_2\right)\right]\left(\frac{a_2}{a_1}\right)^4 \tau\, e^{i(\gamma-4\varphi_{12})} z$$

$$-2\left[\frac{2\mu\beta}{1+\kappa}\left(\frac{k-k_1}{k+k_1}\right)^2\left(\frac{k-k_2}{k+k_2}\right)+\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\frac{\mu_1}{\Gamma_1+\kappa_1}\right.$$

$$\left.\times\left(\beta-\frac{2k}{k+k_1}\beta_1\right)\right]\left(\frac{a_2}{a_1}\right)^2 \tau a_1^4\, e^{i\gamma} z^{-3}$$

$$+2\left[\frac{24\mu\beta\delta_1}{1+\kappa}\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^4+\frac{12\mu_2\delta_1}{\Gamma_2+\kappa_2}\left(\beta-\frac{2k}{k+k_2}\beta_2\right)\left(\frac{a_2}{a_1}\right)^4\right.$$

$$+\frac{4\mu_1\beta}{1+\Gamma_1\kappa}\left(\frac{k-k_1}{k+k_1}\right)\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2+\frac{\mu_1}{\Gamma_1+\kappa_1}\left(\beta-\frac{2k}{k+k_1}\beta_1\right)$$

$$\left.\times\left(\frac{k-k_2}{k+k_2}\right)\left(\frac{a_2}{a_1}\right)^2\right]\tau a_1^6\, e^{i(4\varphi_{12}-\gamma)} z^{-5}. \tag{62}$$

It is to be noted that the stress potentials corresponding to the zero-order of λ in eqns (54) and (55) represent the solutions for a single circular inclusion which are found to agree with the exact results given by Chao and Shen (1995) based upon the method of analytical continuation. The stress potentials in eqns (56)–(62) corresponding to the higher-order terms of the series solution account for the interaction effects between the two inclusions. In the following discussion, we consider either a circular hole or a rigid circular inclusion interacting with an elastic inclusion under a remote uniform heat flux directed from the negative x_1-axis (see Fig. 3). A plane strain condition with $\nu = \nu_1 = \nu_2 = 0.3$ and $a_1 = a_2$ are assumed and the perturbation parameter λ is set to be 0.25 which ensures the good accuracy of the present fourth-order solution.

5.1. *A circular hole interacting with an elastic inclusion*

As our first example, we consider an insulated circular hole ($\Gamma_1 = \beta_1 = k_1 = 0$) interacting with an elastic inclusion ($j = 2$) as shown in Fig. 3. The elastic inclusion is located within the distance $d_{12}/a_1 = 4$ away from a circular hole for the relative inclination φ_{12} ranging from 0° to 180°. Figures 4–6 display the dependence of the normalized hoop stress at point A upon the material constants for different inclinations of an elastic inclusion relative to a circular hole. The effect of the shear modulus of an elastic inclusion upon the hoop stress at point A, when $k_2/k = 2$, $\beta_2/\beta = 1$, can be observed from Fig. 4. It shows that the hoop stress attains the maximum increase or decrease when a circular elastic inclusion becomes a hole ($\Gamma_2 = 0$) for a given relative inclination. This implies that a hole acts as a shield or an antishield depending upon the relative inclination. The effect of the heat conductivity of an elastic inclusion on the hoop stress, when $\Gamma_2 = 5$, $\beta_2/\beta = 1$, is depicted in Fig. 5. The result shows that the two extreme cases of an insulated inclusion ($k_2/k = 0$)

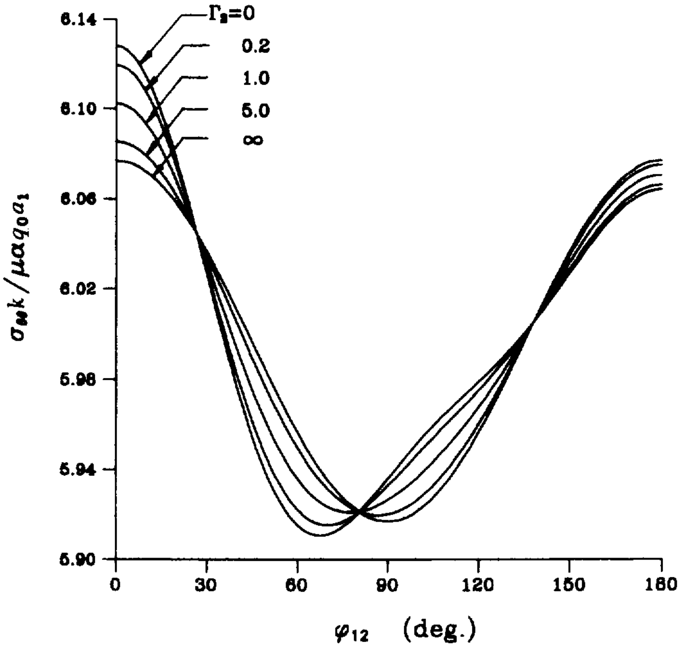


Fig. 4. Variation of hoop stress at point A with Γ_2 and φ_{12} ($k_2/k = 2, \beta_2/\beta = 1$).

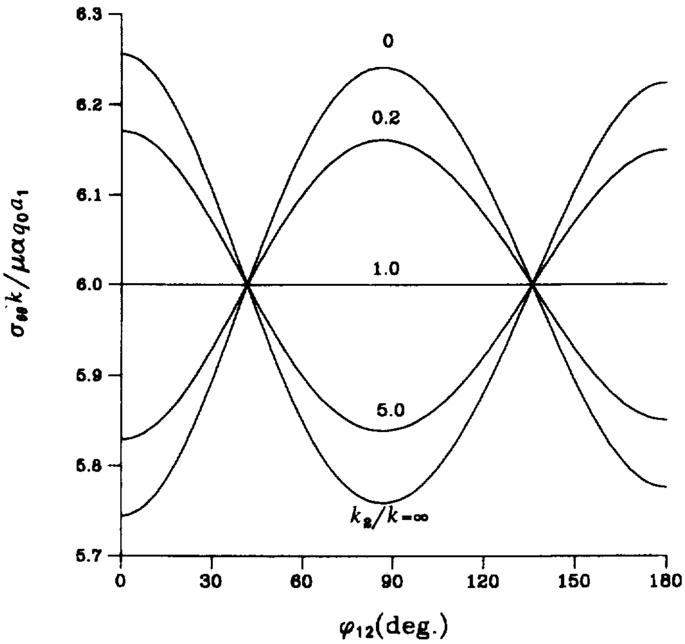


Fig. 5. Variation of hoop stress at point A with k_2/k and φ_{12} ($\Gamma_2 = 5, \beta_2/\beta = 1$).

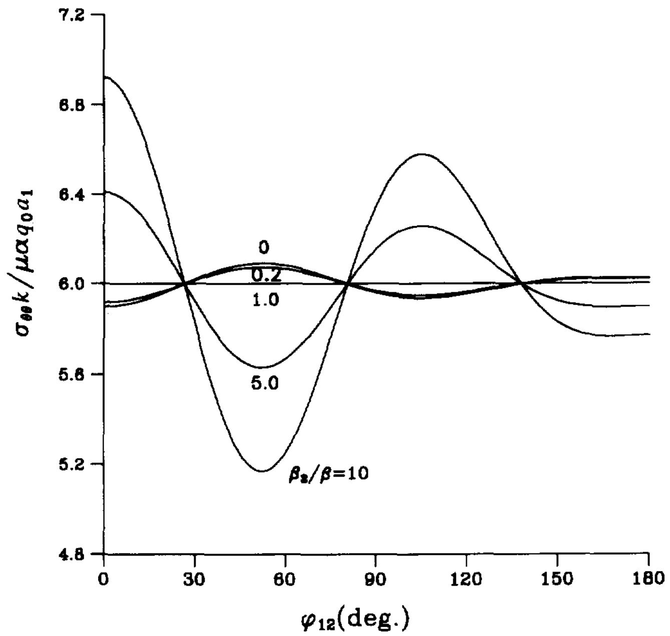


Fig. 6. Variation of hoop stress at point A with β_2/β and φ_{12} ($\Gamma_2 = 2, k_2/k = 1$).

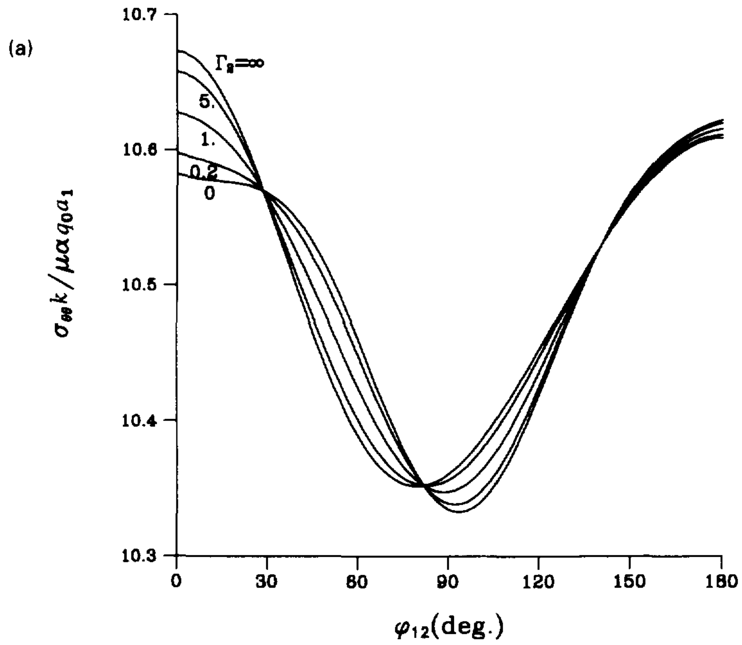
and a fully conductive inclusion ($k_2/k = \infty$) provide the maximum increase or decrease in the hoop stress at point A for a given relative inclination. The influence of the thermal expansion coefficient of an elastic inclusion on the hoop stress at point A, when $\Gamma_2 = 2$, $k_2/k = 1$, can be seen from Fig. 6. It concludes that increasing the thermal expansion coefficient β_2/β of an elastic inclusion may result in an increase of the maximum hoop stress or a decrease of the minimum hoop stress at point A depending on a relative inclination. It is interesting to note that, as shown in Figs 5 and 6, the hoop stress at point A will not be affected by the presence of an elastic inclusion when both the heat conductivity and the thermal expansion coefficient of an elastic inclusion are identical to those of the matrix, i.e. $k_2/k = \beta_2/\beta = 1$. The above conclusion can be further justified by the expression of

$$\mu_2\left(\beta - \frac{2k}{k+k_2}\beta_2\right)$$

as indicated in eqns (59)–(62) which is found to vanish for $k_2/k = \beta_2/\beta = 1$ regardless of the value of μ_2.

5.2. *A rigid circular inclusion interacting with an elastic inclusion*

As our second example, the inclusion ($j = 1$) is assumed to be rigid and unconductive, i.e. $\Gamma_1 = \infty$, $k_1 = \beta_1 = 0$ and an elastic inclusion is located away from a rigid inclusion with the distance $d_{12}/a_1 = 4$. The effect of the shear modulus of an elastic inclusion upon the interfacial stresses at point A, when k_2/k is fixed at 2 and β_2/β is kept at 1, is displayed in Fig. 7(a–c). It can be observed that the interfacial stresses at point A may be increased or decreased depending upon the relative position of an elastic inclusion. The positive tangential stress attains the maximum increase when a rigid inclusion ($\Gamma_2 = \infty$) is placed behind the rigid inclusion ($j = 1$) with $\varphi_{12} = 0°$ as indicated in Fig. 7(a). On the other hand, the negative radial stress is enhanced as a circular hole ($\Gamma_2 = 0$) is located at $\varphi_{12} = 50°$ relative to the rigid inclusion (see Fig. 7(b)). The interfacial shear stress, depicted in Fig. 7(c), is found to change sign as an elastic inclusion moves from $\varphi_{12} = 0°$ to $\varphi_{12} = 180°$. Variations of the interfacial stresses with different values of the heat conductivity of an elastic inclusion are displayed in Fig. 8(a–c). The results indicate that, similar to the hoop



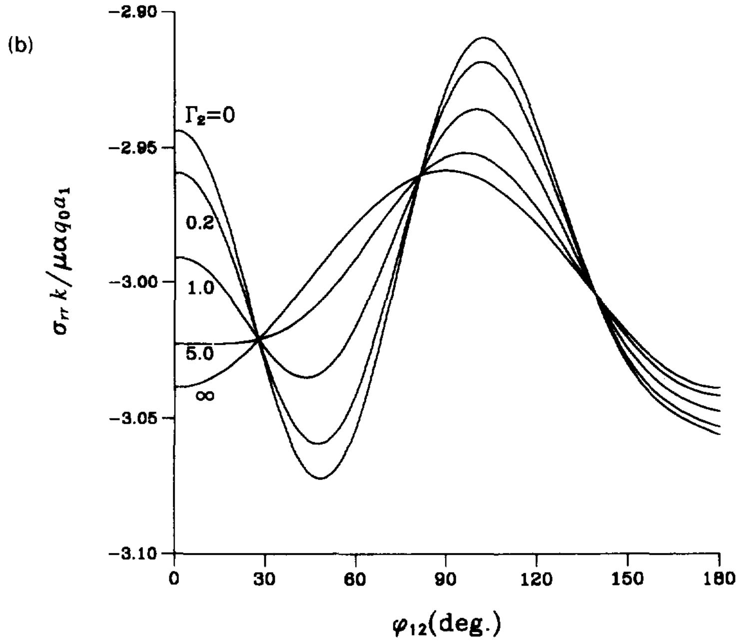


Fig. 7. (a) Variation of interfacial tangential stress at point A with Γ_2 and φ_{12} ($k_2/k = 2, \beta_2/\beta = 1$), (b) variation of interfacial radial stress at point A with Γ_2 and φ_{12} ($k_2/k = 2, \beta_2/\beta = 1$), (c) variation of interfacial shear stress at point A with Γ_2 and φ_{12} ($k_2/k = 2, \beta_2/\beta = 1$). *(Continued overleaf.)*

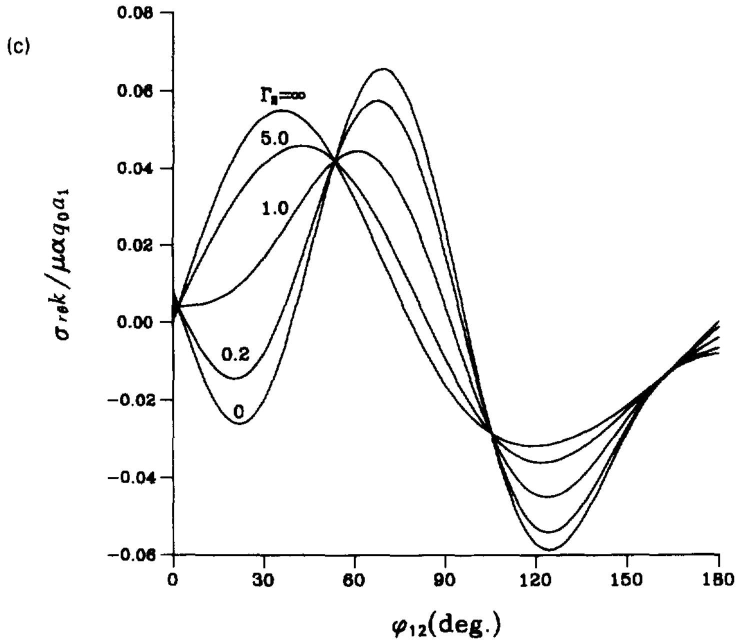


Fig. 7—*Continued.*

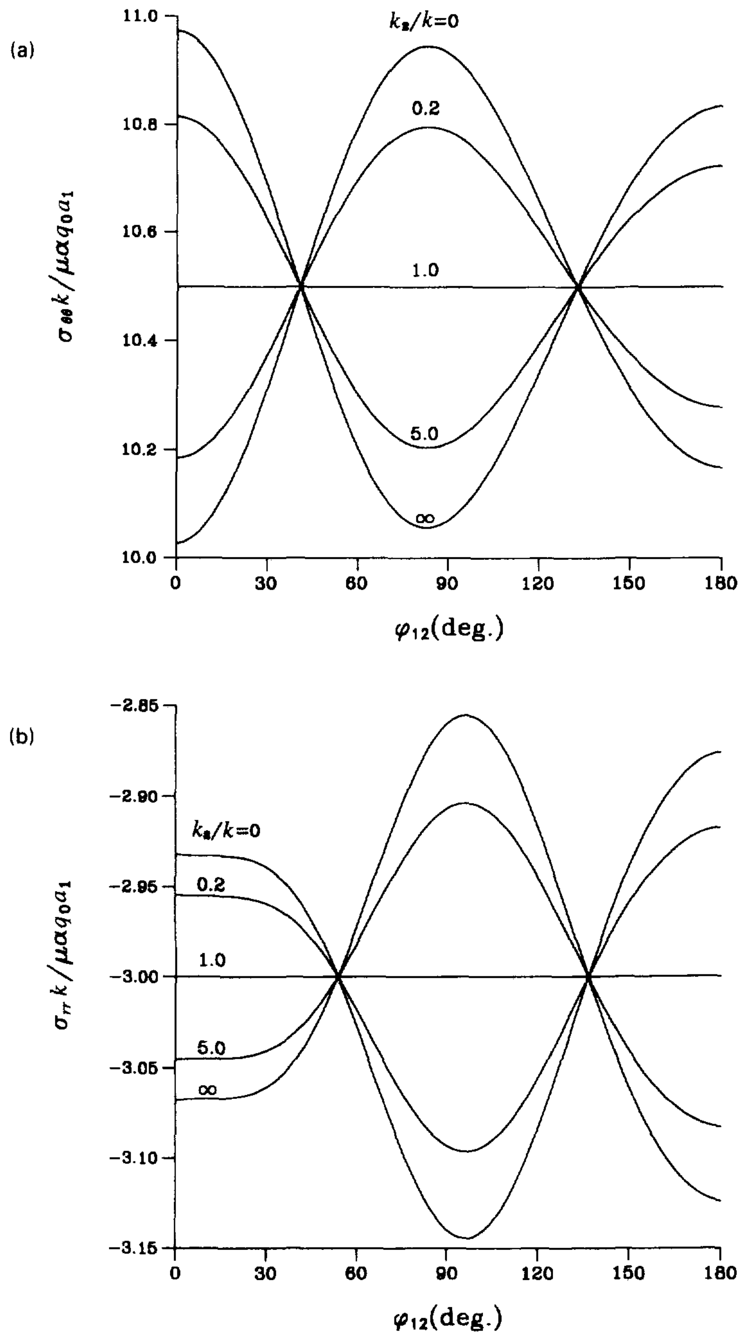


Fig. 8. (a) Variation of interfacial tangential stress at point A with k_2/k and φ_{12} ($\Gamma_2 = 5, \beta_2/\beta = 1$), (b) variation of interfacial radial stress at point A with k_2/k and φ_{12} ($\Gamma_2 = 5, \beta_2/\beta = 1$), (c) variation of interfacial shear stress at point A with k_2/k and φ_{12} ($\Gamma_2 = 5, \beta_2/\beta = 1$). (*Continued overleaf.*)

(c)

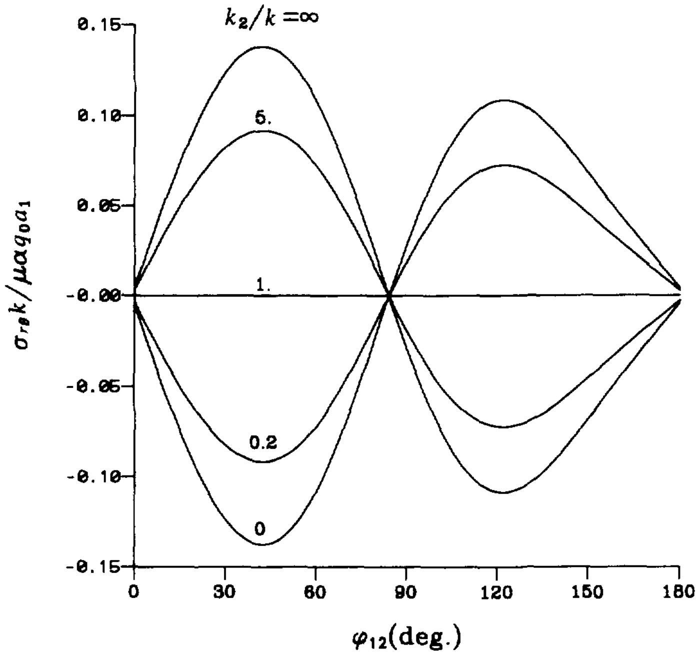
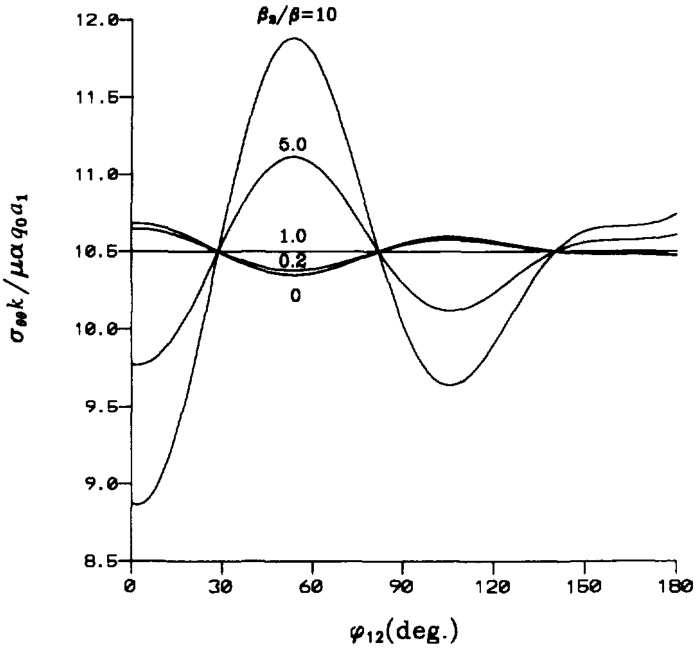


Fig. 8—*Continued.*

(a)



(b)

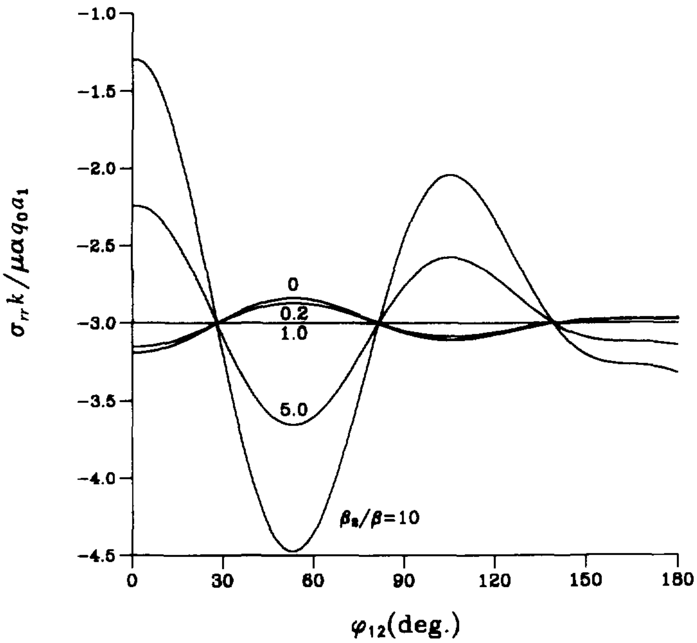


Fig. 9. (a) Variation of interfacial tangential stress at point A with β_2/β and φ_{12} ($\Gamma_2 = 2, k_2/k = 1$), (b) variation of interfacial radial stress at point A with β_2/β and φ_{12} ($\Gamma_2 = 2, k_2/k = 1$), (c) variation of interfacial shear stress at point A with β_2/β and φ_{12} ($\Gamma_2 = 2, k_2/k = 1$).

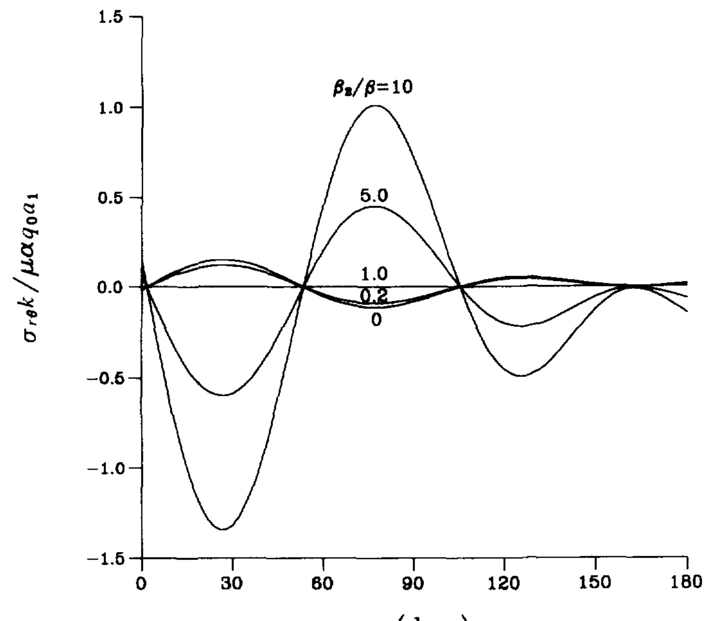


Fig. 9—*Continued.*

stress in Fig. 5, the maximum increase or decrease in the interfacial stresses is provided for the two extreme cases of an insulated inclusion ($k_2/k = 0$) and a fully conductible inclusion ($k_2/k = \infty$). The effect of the thermal expansion coefficient of an elastic inclusion upon the interfacial stresses can be observed in Fig. 9(a–c). Similar to the hoop stress in Fig. 6, increasing of the thermal expansion coefficient of an elastic inclusion will be accompanied by an increase of the maximum interfacial stresses or a decrease of the minimum interfacial stresses for a given relative inclination. Note that, when a uniform heat flux is approached from the negative x_1-axis, all the positive tangential stress and the negative radial stress always prevail at point A regardless of the material constants and the relative position of an elastic inclusion. Furthermore, the interfacial stresses around the rigid inclusion, similar to the hoop stress discussed previously, will not be influenced by the presence of an elastic inclusion as the heat conductivity and the thermal expansion coefficient of an elastic inclusion are the same as those of the matrix.

6. CONCLUSIONS

A general series solution to the thermoelastic multiple inclusion problem is presented via the application of the complex variable theory, the use of the Laurent series expansion and the use of the superposition principle. An approximate closed form solution for the problem of an infinitely extended matrix containing two inclusions is obtained explicitly. Two extreme cases of a circular hole and a rigid circular inclusion interacting with an elastic inclusion are considered. The numerical results of the hoop stress or interfacial stresses at the particular point along the disc are provided in graphic form. Note that the hoop and interfacial stresses along the whole circumference of the disc, not just at the point A, can be also obtained from the present approach. It should be emphasized that the series solution presented in this paper can be derived successively to as many terms as required for the problem that the two inclusions become infinitely close.

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APPENDIX

All the unknown coefficients appeared in eqns (14), (26) and (27) are expressed as power series of λ ($\lambda = a_1/d_{12}$) such that

$$C_{n,j} = \sum_{p=0}^{\infty} C_{n,j}^{(2p)} \lambda^{2p}$$

$$D_{n,j} = \sum_{p=n+1}^{\infty} D_{n,j}^{(2p)} \lambda^{2p}$$

$$M_{n,j} = \sum_{p=0}^{\infty} M_{n,j}^{(2p)} \lambda^{2p}$$

$$K_{n,j} = \sum_{p=0}^{\infty} K_{n,j}^{(2p)} \lambda^{2p}$$

$$F_{n,j} = \sum_{p=n+1}^{\infty} F_{n,j}^{(2p)} \lambda^{2p}$$

$$S_{n,j} = \sum_{p=n}^{\infty} S_{n,j}^{(2p)} \lambda^{2p}.$$

Substituting the above equations into eqns (18), (23), (31), (32), (33), (37), (49) and (50), we have the following relationships expressed as

$$C_{0,j}^{(0)} = d_{12} \tau e^{i\gamma}$$

$$C_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} a_{n,j}^{p,k} D_{p,k}^{(2q)}, \quad (q \geqslant 1)$$

$$D_{n,j}^{(2n+2q)} = \frac{k-k_j}{k+k_j} \bar{C}_{n,j}^{(2q-2)}, \quad (q \geqslant 1)$$

$$M_{n,j}^{(0)} = K_{n,j}^{(0)} = 0$$

$$M_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} a_{n,j}^{p,k} F_{p,k}^{(2q)}, \quad (q \geqslant 1)$$

$$K_{n,j}^{(2q)} = \sum_{k \neq j}^{N} \sum_{p=0}^{q} a_{n,j}^{p,k} S_{p,k}^{(2q)} + \sum_{k \neq j}^{N} \sum_{p=0}^{q-1} b_{n,j}^{p,k} F_{p,k}^{2q}, \quad (q \geqslant 1)$$

$$F_{n,j}^{(2n+2)} = -\delta_j \bar{K}_{n,j}^{(0)}$$

$$F_{n,j}^{(2n+2q)} = -\delta_j \bar{K}_{n,j}^{(2q-2)} - (n+3)\delta_j \bar{M}_{n+2,j}^{(2q-4)} + \frac{2\mu_j \beta}{(n+1)(1+\Gamma_j \kappa)} D_{n+1,j}^{(2n+2q)}, \quad (q \geqslant 2)$$

$$S_{0,j}^{(0)} = S_{1,j}^{(2)} = 0$$

$$S_{0,j}^{(2q)} = -2\gamma_j \,\mathrm{Re}\,[M_{0,j}^{(2q-2)}], \quad (q \geqslant 1)$$

$$S_{1,j}^{(2q+2)} = -\alpha_j \bar{M}_{1,j}^{(2q-2)} + \frac{2\mu\beta}{1+k} D_{0,j}^{(2q)} + \frac{\mu_j\left(\beta - \dfrac{2k}{k+k_j}\beta_j\right)}{\kappa_j + \Gamma_j} \bar{C}_{0,j}^{(2q-2)}, \quad (q \geqslant 1)$$

$$S_{n,j}^{(2n)} = -(n-1)\delta_j \bar{K}_{n-2}^{(0)}, \quad (n \geqslant 2)$$

$$S_{n,j}^{(2n+2q)} = -(n-1)\delta_j \bar{K}_{n-2,j}^{(2q)} - [(n^2-1)\delta_j + \alpha_j]\bar{M}_{n,j}^{(2q-2)} + \frac{2\mu_j\beta}{1+\Gamma_j\kappa} D_{n-1,j}^{(2n+2q-2)}$$
$$+ \frac{2\mu_j}{(1+n)(\Gamma_j+\kappa_j)}\left(\beta - \frac{2k}{k+k_i}\beta_i\right)\bar{C}_{n-1,j}^{(2q-2)}, \quad (n \geqslant 2, q \geqslant 1).$$